Problem 3) Let the two points be (x_1, y_1) and (x_2, y_2). The distance which needs to be maximized (or minimized) is thus given by

$$f(x_1, y_1, x_2, y_2) = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

The first point must lie on the straight line. Therefore, the first constraint is: $g(x_1, y_1, x_2, y_2) = a x_1 + b y_1 = c$.

The second point must lie on the circle. Therefore, the second constraint is $h(x_1, y_1, x_2, y_2) = x_2^2 + y_2^2 = 1$.

We now use the Lagrange multipliers λ_1 and λ_2 to form the following function: $f + \lambda_1 g + \lambda_2 h$. Setting the partial derivatives of this function with respect to x_1, y_1, x_2, y_2 equal to zero, we'll find:

$$\frac{\partial}{\partial x_1}(f + \lambda_1 g + \lambda_2 h) = 2(x_1 - x_2) + \lambda_1 a = 0 \qquad (1)$$

$$\frac{\partial}{\partial y_1}(f + \lambda_1 g + \lambda_2 h) = 2(y_1 - y_2) + \lambda_1 b = 0 \qquad (2)$$

$$\frac{\partial}{\partial x_2}(f + \lambda_1 g + \lambda_2 h) = -2(x_1 - x_2) + 2\lambda_2 x_2 = 0 \qquad (3)$$

$$\frac{\partial}{\partial y_2}(f + \lambda_1 g + \lambda_2 h) = -2(y_1 - y_2) + 2\lambda_2 y_2 = 0 \qquad (4)$$

Equations (1) and (3) may now be solved to yield x_1 and x_2 in terms of λ_1 and λ_2. Similarly, equations (2) and (4) yield the values of y_1 and y_2.

$$x_2 = -\frac{\lambda_1 a}{2\lambda_2}, \quad x_1 = -\frac{\lambda_1 a}{2\lambda_2} - \frac{\lambda_1 a}{2}, \quad y_2 = -\frac{\lambda_1 b}{2\lambda_2}, \quad y_1 = -\frac{\lambda_1 b}{2\lambda_2} - \frac{\lambda_1 b}{2}.$$

Next, we find λ_1 and λ_2 by satisfying the constraints:

$$a x_1 + b y_1 = c \Rightarrow -\tfrac{1}{2}\lambda_1 a^2\left(\tfrac{1}{\lambda_2} + 1\right) - \tfrac{1}{2}\lambda_1 b^2\left(\tfrac{1}{\lambda_2} + 1\right) = c \Rightarrow \lambda_1 (a^2 + b^2) = -\frac{2c\lambda_2}{\lambda_2 + 1}$$

$$x_2^2 + y_2^2 = 1 \Rightarrow \frac{1}{4}\left(\frac{\lambda_1}{\lambda_2}\right)^2 (a^2+b^2) = 1.$$

The above equations now yield λ_1 and λ_2 as follows:

$$\frac{\lambda_1}{\lambda_2} = \pm \frac{2}{\sqrt{a^2+b^2}} \Rightarrow \pm \frac{2\cancel{\lambda_2}}{\sqrt{a^2+b^2}}(a^2+b^2) = -\frac{2c\cancel{\lambda_2}}{\lambda_2+1} \Rightarrow \lambda_2 + 1 = \mp \frac{c}{\sqrt{a^2+b^2}}$$

$$\Rightarrow \begin{cases} \lambda_2 = -1 - \dfrac{c}{\sqrt{a^2+b^2}}, & \lambda_1 = -2\dfrac{\sqrt{a^2+b^2}+c}{a^2+b^2} \\ \lambda_2 = -1 + \dfrac{c}{\sqrt{a^2+b^2}}; & \lambda_1 = +2\dfrac{\sqrt{a^2+b^2}-c}{a^2+b^2} \end{cases}$$

The points (x_1, y_1) and (x_2, y_2) corresponding to the first solution for (λ_1, λ_2) thus turn out to be:

$$\begin{cases} (x_1, y_1) = \left(-\dfrac{\lambda_1 a}{2\lambda_2} - \dfrac{\lambda_1 a}{2}, -\dfrac{\lambda_1 b}{2\lambda_2} - \dfrac{\lambda_1 b}{2}\right) = \left(\dfrac{ac}{a^2+b^2}, \dfrac{bc}{a^2+b^2}\right) \\ (x_2, y_2) = \left(-\dfrac{\lambda_1 a}{2\lambda_2}, -\dfrac{\lambda_1 b}{2\lambda_2}\right) = \left(-\dfrac{a}{\sqrt{a^2+b^2}}, -\dfrac{b}{\sqrt{a^2+b^2}}\right) \end{cases}$$

The second solution for (λ_1, λ_2) yields:

$$\begin{cases} (x_1, y_1) = \left(\dfrac{ac}{a^2+b^2}, \dfrac{bc}{a^2+b^2}\right) \\ (x_2, y_2) = \left(\dfrac{a}{\sqrt{a^2+b^2}}, \dfrac{b}{\sqrt{a^2+b^2}}\right) \end{cases}$$

In both cases, the point (x_1, y_1), located on the straight line $ax+by=c$, is the same. The points (x_2, y_2), however, are on opposite sides of the circle. These three points are all located on the perpendicular dropped from the center of the circle onto the straight line.

Problem 3) Second method) Let the point (x_1, y_1) be located on the straight line $ax + by = c$. Then $y_1 = (c - ax_1)/b$. We will minimize/maximize the distance between (x_1, y_1) and a point (x_2, y_2) on the circle, by assuming at first that x_1 is fixed, then optimizing x_1 afterward. The function to optimize is $f(x_1, x_2, y_2) = (x_2 - x_1)^2 + (y_2 - y_1)^2$, keeping in mind that $y_1 = (c - ax_1)/b$. The constraint on (x_2, y_2) is given by $g(x_2, y_2) = x_2^2 + y_2^2 = 1$. We proceed to optimize the coordinates (x_2, y_2) of the point on the circle using the method of Lagrange multipliers, as follows:

$$\begin{cases} \partial(f + \lambda g)/\partial x_2 = 2(x_2 - x_1) + 2\lambda x_2 = 0 \\ \partial(f + \lambda g)/\partial y_2 = 2(y_2 - y_1) + 2\lambda y_2 = 0 \end{cases} \quad \rightarrow \quad \begin{cases} x_2 = x_1/(1 + \lambda), \\ y_2 = y_1/(1 + \lambda). \end{cases} \tag{1}$$

Next, we find the optimal λ by enforcing the constraint $g(x_2, y_2) = 1$, that is,

$$g(x_2, y_2) = x_2^2 + y_2^2 = (x_1^2 + y_1^2)/(1 + \lambda_0)^2 = 1 \quad \rightarrow \quad \lambda_0 = \pm\sqrt{x_1^2 + y_1^2} - 1. \tag{2}$$

The optimal location of the point on the circle is thus given by

$$(x_2, y_2) = \pm\left(x_1/\sqrt{x_1^2 + y_1^2},\ y_1/\sqrt{x_1^2 + y_1^2}\right). \tag{3}$$

Substituting the above values of (x_2, y_2) in the function $f(x_1, x_2, y_2)$ yields the distance between (x_1, y_1), located on the straight line, and (x_2, y_2), located on the circle, as follows:

$$\begin{aligned} f(x_1, x_2, y_2) &= \left[\left(\pm x_1/\sqrt{x_1^2 + y_1^2}\right) - x_1\right]^2 + \left[\left(\pm y_1/\sqrt{x_1^2 + y_1^2}\right) - y_1\right]^2 \\ &= \left[x_1^2\left(\pm 1 - \sqrt{x_1^2 + y_1^2}\right)^2 + y_1^2\left(\pm 1 - \sqrt{x_1^2 + y_1^2}\right)^2\right]/(x_1^2 + y_1^2) \\ &= \left(\pm 1 - \sqrt{x_1^2 + y_1^2}\right)^2. \end{aligned} \tag{4}$$

Recall that $y_1 = (c - ax_1)/b$, and that, therefore, the above distance is now a function of x_1 only. To find the minimum of the function, its derivative with respect to x_1 must be set to zero.

$$\frac{\mathrm{d}f}{\mathrm{d}x_1} = 2\left(\pm 1 - \sqrt{x_1^2 + y_1^2}\right)\left(-x_1 - y_1\frac{\mathrm{d}y_1}{\mathrm{d}x_1}\right)/\sqrt{x_1^2 + y_1^2} = 0$$

$$\rightarrow\ x_1 + y_1(\mathrm{d}y_1/\mathrm{d}x_1) = 0 \quad \rightarrow \quad x_1 + [(c - ax_1)/b](-a/b) = 0 \quad \rightarrow \quad x_1 = \frac{ac}{a^2 + b^2}. \tag{5}$$

Having found the x-coordinate of the optimal point on the straight line, it is now easy to find the corresponding y-coordinate, namely,

$$y_1 = (c - ax_1)/b = \frac{bc}{a^2 + b^2}. \tag{6}$$

The optimal coordinates of the point on the circle are readily found by substitution into Eq.(3) as

$$(x_2, y_2) = \pm\left(\frac{a}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}}\right). \tag{7}$$

Note that, a perpendicular dropped from the center of the circle onto the straight line, will meet the line at (x_1, y_1) given by Eqs.(5) and (6). The perpendicular crosses the circle at two different locations across a diagonal, as specified by Eq.(7). The point (x_2, y_2) that is closer to (x_1, y_1) will have the shortest distance from the line. The diagonally opposite point on the circle, $(-x_2, -y_2)$, represents a *local* minimum, but not an *absolute* minimum.